Suggested Solutions to: Regular Exam, Fall 2020 Contract Theory January 22, 2021

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Question 1: Private information about both the choice and importance of effort

(a) Suppose the principal wants to induce the outcome $e_A = 1$ and $e_B = 0$. What are the optimal choices of $\underline{t}$ and $\overline{t}$? You are encouraged to show your results by using a graphical analysis.

To induce the outcome $e_A = 1$ and $e_B = 0$, the principal should choose $\underline{t}$ and $\overline{t}$ so as to maximize

$$V = \gamma \left[\pi_1^A \left(\overline{S} - \overline{t}\right) + \left(1 - \pi_1^A\right)\left(\underline{S} - \underline{t}\right)\right] + (1-\gamma)\left[\pi_0^B \left(\overline{S} - \overline{t}\right) + \left(1 - \pi_0^B\right)\left(\underline{S} - \underline{t}\right)\right],$$

subject to the following constraints:

$$\pi_1^A \overline{t} + \left(1 - \pi_1^A\right)\underline{t} - \psi \geq \pi_0^A \overline{t} + \left(1 - \pi_0^A\right)\underline{t}, \tag{IC-A}$$

$$\pi_0^B \overline{t} + \left(1 - \pi_0^B\right)\underline{t} \geq \pi_1^B \overline{t} + \left(1 - \pi_1^B\right)\underline{t} - \psi, \tag{IC-B}$$

$$\gamma\left[\pi_1^A \overline{t} + \left(1 - \pi_1^A\right)\underline{t} - \psi\right] + (1-\gamma)\left[\pi_0^B \overline{t} + \left(1 - \pi_0^B\right)\underline{t}\right] \geq 0, \tag{IR}$$

$$\underline{t} \geq 0 \qquad \text{and} \qquad \overline{t} \geq 0. \tag{LL-L and LL-H}$$

The objective function V is the principal's expected net surplus. She expects the project to be of type A with probability γ, in which case the agent makes an effort and the large surplus is thus realized with probability π_1^A. Similarly, the principal expects the project to be of type B with probability $1-\gamma$, in which case the agent does not make an effort and the large surplus is thus realized with probability π_0^B.

The constraint IC-A (the type A agent's incentive compatibility constraint) ensures that an agent who knows that the project is of type A indeed makes an effort ($e_A = 1$). The left-hand side of the constraint is the agent's expected payoff if knowing that $\tau = A$ and making an effort and thus incurring the effort cost ψ. The right-hand side is the agent's expected payoff if knowing that $\tau = A$, *not* making an effort, and thus not incurring the effort cost. Similarly, the constraint IC-B (the type B agent's incentive compatibility constraint) ensures that an agent who knows that the project is of type B does not makes an effort ($e_B = 0$).

The constraint IR is an ex ante individual rationality (or participation) constraint. It ensures that, in expected terms, the agent's payoff is at least equal to the outside option payoff (which is assumed to be zero). It is the expected payoff that matters, as the agent decides whether to accept or reject the contract offer before having learned the project type.

The constraints LL-L and LL-H are the limited liability constraints that are specified in the question (LL-L and LL-H ensure that the payment is non-negative after a small surplus and large surplus, respectively, has realized).

As there are only two choice variables ($\underline{t}$ and $\overline{t}$), it is possible to solve the problem using a graphical approach. Solving the objective function and the constraints for $\overline{t}$ (which will be measured on the vertical axis of the graph), we obtain:

$$\overline{t} = \frac{(1-\widehat{\pi})\,\underline{S} + \widehat{\pi}\overline{S}}{\widehat{\pi}} - \frac{V}{\widehat{\pi}} - \frac{1-\widehat{\pi}}{\widehat{\pi}}\underline{t} \qquad \text{(P's indiff curve)}$$

$$\overline{t} \geq \frac{\psi}{\pi_1^A - \pi_0^A} + \underline{t}, \qquad \text{(IC-A)}$$

$$\overline{t} \leq \frac{\psi}{\pi_1^B - \pi_0^B} + \underline{t}, \qquad \text{(IC-B)}$$

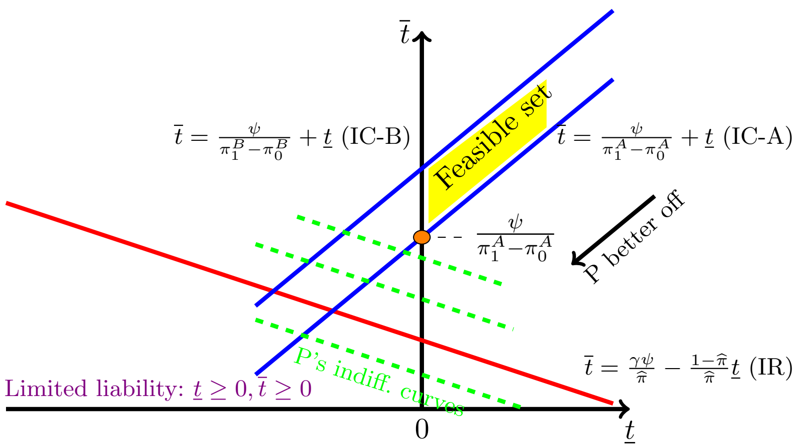
$$\overline{t} \geq \frac{\gamma\psi}{\widehat{\pi}} - \frac{1-\widehat{\pi}}{\widehat{\pi}}\underline{t}, \qquad \text{(IR)}$$

where

$$\widehat{\pi} \stackrel{\text{def}}{=} \gamma\pi_1^A + (1-\gamma)\pi_0^B.$$

In the figure below, the constraints are graphed in the $(\underline{t}, \overline{t})$-space. In addition, a few of P's indifference curves (in green) are shown. The feasible choice set is indicated in yellow, and P is better off in the south-west direction. This means that the optimum must be in the south-west corner of the feasible set (at the orange dot), where IC-A and LL-L bind (and all other constraints are lax). Solving those two binding constraints for the two choice variables yield the following solution to the problem:

$$(\underline{t}, \overline{t}) = \left(0, \frac{\psi}{\pi_1^A - \pi_0^A}\right). \qquad (1)$$



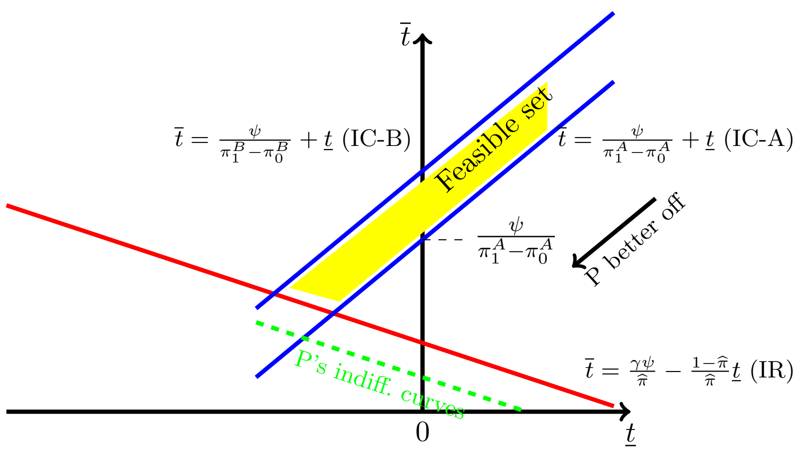
Conclusion: The unique optimal contract is $(\underline{t}, \overline{t}) = (0, \frac{\psi}{\pi_1^A - \pi_0^A})$. At the optimum, the constraints IC-A and LL-L bind whereas IR, IC-B, and LL-H are lax.

(b) Suppose there are no limited liability constraints in this model (meaning that any $\underline{t} \in \mathbb{R}$ and $\bar{t} \in \mathbb{R}$ are allowed). With this assumption, can the principal induce the outcome $e_A = 1$ and $e_B = 0$ without giving away rents to the agent? Explain why or why not. Also explain any differences in the reasoning and the conclusions relative to the model we studied in the course (i.e., the 2x2 moral hazard model with a risk neutral agent and with known probabilities of a large surplus, with and without effort).

If there are no limited liability constraints, the figure we used is modified—see the new figure below. The feasible set is now expanded and stretches all the way down to the boundary of A's IR constraint. The slope of the IR constraint equals the slope of the indifference curves. Hence, the optimal solutions are all those combinations of $\underline{t}$ and $\bar{t}$ such that IR binds and both IC-A and IC-B are satisfied:

$$\bar{t} = \frac{\gamma\psi}{\widehat{\pi}} - \frac{1-\widehat{\pi}}{\widehat{\pi}}\underline{t} \quad \text{and} \quad \underline{t} \in \left[-\frac{\psi\left[\gamma(\pi_1^A - \pi_1^B) + \pi_0^B\right]}{\pi_1^B - \pi_0^B}, -\frac{\psi\left[\gamma\pi_0^A + (1-\gamma)\pi_0^B\right]}{\pi_1^A - \pi_0^A}\right] \tag{2}$$

This means that the principal can indeed induce the outcome $e_A = 1$ and $e_B = 0$ without giving away any rents (we know this as the IR constraint binds at the optimum). The way in which the principal can achieve this is, essentially, the same as in the model that we studied in the course—namely, by creating an appropriately large difference between the payments after a large and small surplus; this makes the agent face effort incentives that are of the right strength for her to choose what the principal wants her to choose. The only difference is that, here, there is a limit on how small the payment $\underline{t}$ can be made, due to the constraint IC-B. If the difference between the payments is made too large, then the agent does not have an incentive to refrain from making an effort when knowing that the project is of type B. That is, in this model with private information about the project type and no limited liability constraints, the principal can still induce the agent to make the desired effort choices, without giving away any rents; however, the payments in the contract must now be more fine tuned—the difference between the payments must be neither too small nor too large.



Question 2: Adverse selection in a competitive insurance market

(a) Solve as much as you need of the above problem to show how $\underline{u}_N^{SB}$ relates to $\underline{u}_A^{SB}$, and how $\overline{u}_N^{SB}$ relates to $\underline{u}_A^{SB}$. You do not need to show that the second-order condition is satisfied (and if you nevertheless do that, you will not get any credit).

For reasons that will be discussed in part (b), we should suspect that neither one of the IC constraints binds at the optimum. Thus, let us guess that they are both lax at the optimum (we must check this later). There is now only one constraint left—namely, the profit constraint—and the Lagrangian can be written as

$$\mathcal{L} = \nu \left[(1-\underline{\theta}) \underline{u}_N + \underline{\theta} \underline{u}_A \right] + (1-\nu) \left[\left(1-\overline{\theta}\right) \overline{u}_N + \overline{\theta} \overline{u}_A \right]$$
$$- \lambda \left\{ \nu \left[(1-\underline{\theta}) h(\underline{u}_N) + \underline{\theta} h(\underline{u}_A) \right] + (1-\nu) \left[\left(1-\overline{\theta}\right) h(\overline{u}_N) + \overline{\theta} h(\overline{u}_A) \right] - K \right\},$$

where $\lambda \geq 0$ is the shadow price associated with the profit constraint. The first-order conditions are:

$$\frac{\partial \mathcal{L}}{\partial \underline{u}_N} = \nu (1-\underline{\theta}) - \lambda \nu (1-\underline{\theta}) h'(\underline{u}_N) = 0 \Leftrightarrow \lambda h'(\underline{u}_N) = 1,$$

$$\frac{\partial \mathcal{L}}{\partial \underline{u}_A} = \nu \underline{\theta} - \lambda \nu \underline{\theta} h'(\underline{u}_A) = 0 \Leftrightarrow \lambda h'(\underline{u}_A) = 1,$$

$$\frac{\partial \mathcal{L}}{\partial \overline{u}_N} = (1-\nu) \left(1-\overline{\theta}\right) - \lambda (1-\nu) \left(1-\overline{\theta}\right) h'(\overline{u}_N) = 0 \Leftrightarrow \lambda h'(\overline{u}_N) = 1,$$

$$\frac{\partial \mathcal{L}}{\partial \overline{u}_A} = (1-\nu)\overline{\theta} - \lambda (1-\nu) \overline{\theta} h'(\overline{u}_A) = 0 \Leftrightarrow \lambda h'(\overline{u}_A) = 1.$$

It follows immediately from these four first-order conditions that $\lambda > 0$ and that $h'(\underline{u}_N) = h'(\underline{u}_A) = h'(\overline{u}_N) = h'(\overline{u}_A)$. As h' is a strictly increasing function, this in turn implies that the four arguments must also equal each other: $\underline{u}_N = \underline{u}_A = \overline{u}_N = \overline{u}_A$.

It remains to check whether the two IC constraints are satisfied at the candidate optimum that we found. If indeed $\underline{u}_N = \underline{u}_A = \overline{u}_N = \overline{u}_A$, then each one of the two IC constraints simplify to $0 \geq 0$, which is trivially satisfied. We can thus conclude that the candidate optimum is indeed the true optimum. Thus, at the second-best optimum, there is full insurance for both types: $\underline{u}_N = \underline{u}_A$ and $\overline{u}_N = \overline{u}_A$.

(b) In the course, we studied a model of a *monopoly* insurance market with adverse selection. Discuss how the economic logic of that model and the one described in this question differs (if you think there is anything in the economic logic that differs). In particular, in each one of the two models, what is the tradeoff that the principal faces, and how can that information help us understand how the optimal contracts look like in the two models?

- In the monopoly model that we studied in the course, there is a tradeoff between rent extraction and allocative efficiency. One the one hand, it is in the insurance company's interest to create allocative efficiency and thereby make the size of the pie of resources as large as possible. On the other hand, it is also in the company's interest to avoid giving away rents to the agent, as the firm wants to maximize its profits. There is a conflict between these two objectives, as too much allocative efficiency creates problems with incentive compatibility, which can be avoided by giving away rents to the high-demand agent type.

- In the present model with perfect competition, the principal of the model is not a profit-maximizing company. Instead we can interpret the principal as a representative of the collective of consumers (as the objective function is given by the expected consumer payoff). This means that the principal no longer has an interest in extracting as much rents as possible from the consumer. Thus, the rent-extraction aspect of the tradeoff that was present in the monopoly model is no longer there. The principal is happy with choosing full allocative efficiency and does not mind that this also means giving away rents.

- The tradeoff that is present in the monopoly model gives rise to the result that the low-demand type underinsures. In the competitive model there is no tradeoff, which means that we should expect the optimum to involve full insurance, which is indeed what we found in the above analysis.